Advertising and Sealed Bid Auctions in a Transshipment Game

by

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Abstract

In an earlier paper [7], we suggested employing the sealed-bid auction as a format for modelling the marketing of a wide range of heterogeneous products, including the marketing of most brand names. The formation of prices in such markets is described *as if* it resulted from an actual sealed-bid auction, with buyers inspecting the goods offered by sale by the various manufacturers, and making sealed bids which later are opened by a hypothetical auctioneer. While surely putative, this model nevertheless seems superior to the standard assumptions of perfect competition where a homogeneous commodity is transacted and all buyers offer identical prices.

Extending the use of the sealed bid auction, we here consider cases where the manufacturers or distributors promote their sales by advertising. While the maximum quantity demanded by each customer is fixed and given, his bids on the various brands are determined by the advertising and other promotional efforts. The bid response function for each consumer is supposed to be given and known. The preceding vertical production and distribution chain, from the supply of raw materials and primary commodities, via the successive processing, manufacturing and distribution of the product is described as a transshipment auction game.

We formulate a complementarity model that in one step solves this entire logistics system, including a possible formation of coalitions between suppliers and/or distributors, the market prices of the various brands, the optimal distribution of the product, and the bids of the consumers. A numerical example is provided and the solution routine is discussed.

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1. Introduction.

In spite of the fact that most goods and services in a modern economy are heterogeneous, that is, display features which have characteristics associated with the manufacturer (such as brand names, trademarks, etc.), and which compete in individual "niches" of the market, economic theory has had difficulties in finding an attractive theoretical format for capturing such realities. Too often economists simply assume that such goods are traded in markets of perfect competition, even though the assumption that each market participant is a price taker manifestly does not apply. A general analytic format encompassing most of the modern forms of marketing of heterogeneous goods, which is mathematically comprehensive, simple and straightforward, has not heretofore been available.

The mathematical format of a sealed bid auction is suggested here as the basis for a new market paradigm which may provide what is desired. It is well known that the sealed bid auction is frequently used for the pricing of many commodities, such as furs, real estate, and oil leases. But the important observation is that the pricing of heterogeneous goods in the modern economy can be perceived as occurring as if it resulted from a sealed bid auction. That is, the sealed bid auction model will be proposed here as a mathematical model for determining the pricing of heterogeneous goods quite generally, whether or not they are actually sold under conditions of a formal sealed bid auction.

In a sealed bid auction the sellers arrive at the market with the goods they wish to sell, each having its brand names and advertising images. Each potential buyer inspects these goods and enters sealed bids for the goods of the sellers. Naturally each buyer will attach high bids to the goods whose design, packaging, and product image he likes, and low bids to other goods that seem to him to have less quality, less appeal or shoddy appearance. The bid prices thus indirectly reflect the buyers' relative estimation of the various heterogeneous goods offered for sale in the market. It is also assumed that the buyer announces the maximum quantity of each good that he is willing to buy, that it, his demand.

Each seller in the auction is permitted to state his reservation price, which is the lowest price at which he is willing to sell his good. Since his goods are on display, it is also clear what the maximum among of each good is that he is offering for sale, that is, his supply.

The auctioneer (which in our case will be a mathematical program) then determines the market price of the good of each seller, and the actual exchanges of goods between sellers and buyers.

The market price paid by a given buyer cannot exceed his bid price; in case it is less than his bid price, the difference between the bid and actual price is his "buyer surplus." Buyers whose bids are low may receive none or only part of their demand. But in no case will a buyer be required to buy more than his stated demand.

Similarly, the market price received by a given seller cannot be less than his reservation price; in case it is greater than his reservation price, the excess is his "seller surplus". Sellers who have high reservation prices may not sell all of the good they offer. In no case will a seller be required to sell more than his supply.

The optimal price for each good, which is to be found by solving a mathematical program, is not uniquely determined. There exists a set of alternate optimal solutions for prices, called the *core* of the auction, from which one single solution must be chosen to determine the actual prices. One such solution is very favorable to the sellers, and consequently unfavorable to the buyers, and there is another solution which is favorable to the buyers and unfavorable to the sellers. In this paper we shall determine, characterize and show how to compute these two extreme solutions, called the MaxSeller-MinBuyer surplus, and MinSeller-MaxBuyer surplus, solutions, respectively. Clearly, if the sellers organize the auction, the MaxSeller-MinBuyer solution will probably be chosen. But if the buyers organize the auction the MinSeller-MaxBuyer solution will probably be chosen. If some other coalition of buyers or sellers controls the auction a solution favorable to them might be chosen.

We define the *fair division solution* to be the average of these two extreme solutions, and suggest that it may be a suitable choice for the outcome of auctions in which neither the buyers nor the sellers nor any other coalition of buyers and/or sellers have control over the auction mechanism that determines prices.

In the standard auction model, the sellers passively bring their goods to the market and accept whatever price results from the auction. Here, we shall explore the possibility of permitting the sellers to conduct an active policy of marketing and advertising, with the purpose of strengthening market demand and improving their net proceeds from the auction. More specifically, we shall equip each seller with a given advertising budget. The detailed advertising expenditures may be targeted particular consumer groups. Successful advertising increases the awareness of a consumer or a consumer group of the product, and enhances the bid price offered by the consumer or consumer group. Indirectly, the advertising will then also raise the auction price received by the seller.

To simplify matters, we shall assume throughout that there exists a unique "channel" of advertising (such as TV or radio advertisements) and that the response of each consumer to a dollar of advertising expenditures by a given seller is given and known. In any given application, the actual relationships will no doubt be more complex. There exist several channels of advertising open to each seller. The response of consumers to a minute of TV advertising varies by the hour of the day, and by demographical characteristics. The price of a minute of TV advertising depends upon the time slot chosen for airing, and so on. The mathematical generalizations required to handle circumstances like these, and others, should nevertheless be fairly straightforward and need not delay us here.

The basic model is outlined in Section 2. It extends the standard model of sealed bid auction, see Shapley and Shubik [3] and Thompson [4,5,6]. Section 3 defines and investigates the core. Section 4 provides a simple numerical example. Section 5 turns to a more elaborate

model featuring an entire production -distribution system. Section 6 gives a numerical example. Section 7 contains the solution to a coalition of players for the latter example.

2. The Basic Model

Consider a sealed-bid auction organized as follows. There are m suppliers. They all market the same basic product, such as coffee or oranges. Each supplier offers a distinct variety or brand of the product, and advertises and markets it under a characteristic brand name. Thus the good becomes heterogeneous, that is, each supplier offers a single uniform product, but the market features many different brands. Consumers of the product associate each brand with its particular supplier. Each supplier determines his reservation price, that is, the lowest (auction) price at which he is willing to offer the product on the market.

There are r consumers. Each consumer has a given maximal demand for the product, regardless of price. Each consumer inspects all brands that are offered for sale and writes down on a piece of paper a bid for each brand, reflecting its apparent quality, and the attractiveness of the brand. The bids are sealed. All sealed bids are opened at the same time and the auction price to be obtained by each seller is determined. If a consumer pays less than his bid price, he obtains a buyer surplus; if a supplier receives more than his reservation price he, in the same way, obtains a seller surplus.

For a general survey of the economic analysis of auctions, see McAfee and McMillan [2]. For an introduction to the standard mathematical model of a sealed bid auction, see Thompson and Thore [8], Chap. 2.

Each supplier has an advertising budget and may spend advertising dollars targeting one, several, or all consumers. The purpose of advertising is to enhance the consumers' bid prices, and ultimately, the auction price received by the supplier. The response of each consumer's bid to one dollar of advertizing is given and known.

The problem at hand is to determine the advertising strategy of each supplier, and the resulting auction price received by each supplier.

Use the following notation:

I = {1,..., m} is a set of suppliers; i is a generic member;
K = {1,..., r} is a set of consumers (buyers); k is a generic member.

and also the following parameters:

a_i is the supply of supplier i; it is assumed that $a_i > 0$;
d_{ik} is the bid placed by consumer k on the wares of supplier I, in the absence of any advertising;
d_{ik}' is the response of the bid by consumer k to one dollar of advertising expenditure spent by supplier i and targeted consumer k

s_i	is the reservation price of supplier i for his good;
b_k	is the demand of consumer k; it is assumed that $b_k > 0$;
p_k	is the value to consumer k of one generic unit of unsatisfied demand;
A_i	is the total advertising budget of supplier i;

and also the following unknowns

x_{ik}	is the quantity sold by supplier i to consumer k;
z_i	is the unsold supply of supplier i;
α_{ik}	advertising expenditure by supplier i, in dollars, targeted consumer k;
t_k	is the unsatisfied demand of consumer k;
u_i	the auction price of the wares of supplier i;
w_k	the buyer's surplus obtained by consumer k;
q_i	the imputed value to supplier i of one additional dollar of advertising

Note that the total bid value placed by consumer k on the wares of supplier i equals $(d_{ik} + d_{ik}'\alpha_{ik})\, x_{ik}$.

With these definitions, the basic model consists of relations (1) - (8) below.:

$$\sum_K x_{ik} + z_i = a_i \quad \text{and} \quad u_i z_i = 0 \quad \text{for } i \varepsilon I \qquad (1)$$

$$\sum_I x_{ik} + t_k = b_k \quad \text{and} \quad w_k t_k = 0 \quad \text{for } k \varepsilon K \qquad (2)$$

$$\sum_K \alpha_{ik} \leq A_i \quad \text{and} \quad (A_i - \sum_K \alpha_{ik})\, q_i = 0 \quad \text{for } i \varepsilon I \qquad (3)$$

Relations (1) state that the total sales of a given supplier cannot exceed his available supply. But if they fall short of the available supply, the auction price fetched by this supplier must have dropped to zero. Relations (2) state that the total sales of all brands to a given consumer cannot exceed his demand. But if they do fall short of his demand, his surplus must vanish. Relations (3) state that the total advertising expenditure of each supplier cannot exceed his available advertising budget. But if they fall short of the budget, the imputed value of one additional dollar of advertising is zero.

Note that z_i, the unsold wares of supplier i, are valued at s_i, which is his reservation price. Similarly, the unsatisfied demand t_k, of consumer k, is valued at $-p_k$, since this quantity represents the value to the consumers of unfulfilled demand. (Later we will discuss how to choose p_k.)

Further,

$$u_i + w_k \geq d_{ik} + d_{ik}'\alpha_{ik}$$

$$\text{and } x_{ik}(u_i + w_k - d_{ik} - d_{ik}'\alpha_{ik}) = 0 \qquad \text{for } i\varepsilon I,\ k\varepsilon K \qquad (4)$$

$$q_i \geq d_{ik}'x_{ik} \qquad \text{and } \alpha_{ik}(q_i - d_{ik}'x_{ik}) = 0 \qquad \text{for } i\varepsilon I,\ k\varepsilon K \qquad (5)$$

$$u_i \geq s_i \qquad \text{and } z_i(u_i - s_i) = 0 \qquad \text{for } i\varepsilon I \qquad (6)$$

$$w_k \geq -p_k \qquad \text{and } t_k(w_k + p_k) = 0 \qquad \text{for } k\varepsilon K \qquad (7)$$

Relations (4) state that the bid price of a consumer ($d_{ik} + d_{ik}'\alpha_{ik}$) offered for the wares of supplier i can never exceed the sum of the auction price and his surplus. Moreover, if the consumer buys a positive quantity from that supplier, the bid must exactly equal the sum of the price and his surplus. That, of course, is just the definition of the buyer surplus. Relations (5) state that the imputed value of one additional dollar of advertising cannot fall short of the total bid value that such a dollar produces. Moreover, if advertising is positive, the imputed value must equal that bid value. Relations (6) state that the auction price cannot fall short of the reservation price of the supplier. If a supplier returns from the auction with some unsold quantity, the auction price must have dropped to the reservation price. Relations (7) state that the surplus of a consumer is greater than or equal to the negative of the value he places on not buying one unit of the good. In the case that he is unable to buy all he wants, his surplus is equal to the negative of that value.

Finally, there are the following nonnegativity conditions:

$$x_{ik}, \alpha_{ik}, z_i, t_k \geq 0 \qquad \text{for } i\varepsilon I,\ k\varepsilon K \qquad (8)$$

The unknowns u_i, $i\varepsilon I$ and w_k, $k\varepsilon K$ are unrestricted in sign.

3. The Core

Although it is not obvious, there are many solutions to the dual problem. It can be shown that the set of all dual solutions is a bound polyhedral convex set C called the *core*, see [3], [4]. Reference [5] gives an algorithm for finding all the extreme points of the core. Here we shall discuss only two, the MaxSeller-MinBuyer and the MinSeller-MaxBuyer surplus extreme points. We assume throughout this section that the coefficients a_i and b_k are integers for i in I and k in K.

The terms "max" and "min" as used in "MaxSeller", "MinBuyer" etc. refer to various partitionings of the total bid value $\sum_I \sum_K (d_{ik} + d_{ik}'\alpha_{ik})\, x_{ik}$.

In order to find the MaxSeller-MinBuyer surplus point we perturb the b_k coefficients slightly be replacing them by $b_k + \varepsilon$, where ε is a small number, such as .001, .0001, etc. It is clear that this substitution will affect the solution of $x_{ik}, \alpha_{ik}, z_i, t_k$ for $i\varepsilon I$, $k\varepsilon K$ only slightly, and that scientific rounding of its solution will give back the same answers that were found

without the perturbation. Note that the relations (4) - (7) are completely unaffected by this change. Analogous to [4] it may be shown that this solution, which is the Max Seller - MinBuyer solution, also minimizes each consumer's surplus individually, and maximizes each seller's surplus individually.

In the same manner, if we replace the coefficients a_i by $a_i + \varepsilon$ (after deleting the previous perturbation of the b_k coefficients). Analogous to [4] it may be shown that this solution, which is the MinSeller-MaxBuyer solution, also minimizes each seller's surplus individually, and maximizes each buyer's surplus individually.

Another property of these two extreme points which is proved in [5] is that the MinSeller-MaxBuyer and MinBuyer-MaxSeller solutions are the farthest apart of any pair of points in the core. It is remarkable that by solving just two simple linear programs, these two important extreme points of the core can be found.

The problem of computing *all* of the extreme points of the core is formidable since it can be shown, see [5], that their number increases exponentially with m and n. Hence it is unlikely that anyone will compute all of them, except for small values of m and n.

If the primal problem (1) is solved without any perturbations, some random extreme point of the core will be chosen, and precisely which extreme point chosen will depend upon the pivot selection rule embodied in the linear programming code used to solve the problem. We call this "letting the computer select" the extreme point.

In real life, auctions are organized by people, frequently either the buyers or sellers themselves. In the case that the buyers organize the auction, they will probably choose the MinSeller-MaxBuyer solution. But in the case that the sellers are the organizers, the MaxSeller-MinBuyer extreme point will probably be chosen. There are literally thousands of different auction mechanisms in use in the world today, and it would be an interesting project to determine how the solutions to the auction are found in actual practice.

Some auctions are arranged by neutral organizations, such as a government agency, a non profit organization, etc. In these cases the *fair division solution,* which is an interior point of the core, may well be a good solution to use. To compute the fair division solution, all that is needed is to average the values of the MaxSeller-MinBuyer and MinSeller-MaxBuyer solutions of each auction participant. When this solution is used, it can be shown that every buyer and every seller receives exactly half way between the maximum and the minimum he can receive from any core solution point. Again this is a remarkable property!

4. A Numerical Example

Consider an economy in which there are 3 suppliers and 4 final consumers. We first list the supplies, reservation prices, and the advertising budgets (in dollars).

Supplier	Supplies	Reservation price	Advertising budget
1	80	18	10
2	100	18	10
3	60	18	10

The bids of the consumers in the absence of any advertising, and their demands are exhibited in the next table.

Supplier	Bid by consumer 1	Bid by consumer 2	Bid by consumer 3	Bid by consumer 4
1	22	18	21	25
2	25	23	22	20
3	19	24	18	24
Demands	30	60	80	70

Finally, we list the response of the bid prices to each dollar of advertising expenditure.

Advertising by	Additional bid by consumer 1	Additional bid by consumer 2	Additional bid by consumer 3	Additional bid by consumer 4
supplier 1	0.5	0.5	0.5	0.5
supplier 2	0.1	0.1	0.1	0.1
supplier 3	0.2	0.2	0.2	0.2

All consumer surpluses are required to be nonnegative ($p_k = 0$ for all k).

The solution advertising expenditures are exhibited first. Note that each supplier entirely spends his advertising budget targeting a single consumer.

Advertising by supplier	targeting consumer 1	targeting consumer 2	targeting consumer 3	targeting consumer 4
1	0	0	0	10
2	0	0	10	0
3	0	10	0	0

The resulting quantities sold are listed next. Each supplier sells all his supplies. Each consumer fills his demands.

From supplier	to consumer 1	to consumer 2	to consumer 3	to consumer 4
1	0	0	10	70
2	30	0	70	0
3	0	60	0	0

The solution bids (including the responses to advertising) are listed below. As a result of the advertising, the bid received by supplier 1 from consumer 4 has now been raised from 25 to 30. The bid received by supplier 2 from consumer 3 has been raised from 22 to 23. The bid by supplier 3 from consumer 2 has been raised from 24 to 26.

Bids placed on goods offered by supplier	Bid by consumer 1	Bid by consumer 2	Bid by consumer 3	Bid by consumer 4
1	22	18	21	30
2	25	23	23	20
3	19	26	18	24

As explained, the solution for the auction prices depends on the perturbations applied to the supplies and demands. In the tables below, Solution 1 means the MinSeller-MaxBuyer solution (obtained by perturbing supplies) , Solution 2 means the MaxSeller-MinBuyer solution (obtained by perturbing demands) , and FDS means the fair division solution.

Auction price quoted for the wares of...	Solution 1	Solution 2	FDS
supplier 1	18	21	19.5
supplier 2	20	23	21.5
supplier 3	18	26	22

and the surpluses are:

Surplus obtained by...	Solution 1	Solution 2	FDS
consumer 1	5	2	3.5
consumer 2	8	0	4
consumer 3	3	0	1.5
consumer 4	12	9	10.5

5. Advertising in an entire transportation - distribution system with auctions.

Consider now an entire economic logistics system organized as follows. There are m supply points or origins; at each supply point there is a known supply of some homogeneous commodity, such as coffee beans. The commodity is shipped to n processing plants or destinations, each owned by different wholesalers. At a given destination the wholesaler processes the commodity it receives and markets it under its own brand name. In the case of

coffee, each wholesaler roasts and blends his own coffee and packages it with his brand name. Thus the good becomes heterogeneous, that is, each wholesaler sells a single uniform product, but the market for all of these products features many different brands. Consumers of the product associate each brand with its particular wholesaler.

The wholesaler is assumed to be a price taker and no provision is made for him to have a reservation price. (This could easily be added to the model if desired.) We use expanded and modified notation as follows. First, there are the three sets of economic agents:

I	$=\{1,...,m\}$ is a set of m suppliers; i is a generic member;
J	$=\{1,...,n\}$ is a set of n wholesalers (destinations); j is a generic member;
K	$=\{1,...,r\}$ is a set of r consumers (buyers); k is a generic member.

The parameters to be used are listed next. Note that this time we let the wholesalers be endowed with given advertising budgets (rather than the suppliers).

a_i	is the supply of supplier i; it is assumed that $a_i > 0$;
c_{ij}	is the unit shipping cost along arc (i,j);
e_{jk}	is the unit shipping cost on arc (j,k);
h_j	is the unit processing and packaging cost of each wholesaler;
d_{jk}	is the bid placed by consumer k on the wares of wholesaler j;
s_i	is the reservation price of supplier i for his good;
b_k	is the demand of consumer k; it is assumed that $b_k > 0$;
p_k	is the value to consumer k of one generic unit of unsatisfied demand.
A_j	is the total advertising budget of wholesaler j;

There are the following unknowns

x_{ij}	is the quantity shipped along arc (i, j), to be determined;
y_{jk}	is the quantity shipped on arc (j,k);
z_i	is the unsold supply of supplier i;
t_k	is the unsatisfied demand of consumer k;
α_{jk}	advertising expenditure by wholesaler j, in dollars, targeted consumer k;
t_k	is the unsatisfied demand of consumer k;
u_i	the imputed value of one unit of the commodity at origin i;
v_j	the auction price of the wares of wholesaler j;
w_k	the buyer's surplus obtained by consumer k;
q_j	the imputed value to wholesaler j of one additional dollar of advertising

With these definitions, consider the model given by (9) - (18) below.

$$\sum_J x_{ij} + z_i = a_i \qquad \text{and} \qquad u_i z_i = 0 \qquad \text{for } i\varepsilon I \qquad (9)$$

$$-\sum_I x_{ij} + \sum_K y_{jk} \leq 0 \qquad \text{and}$$

$$v_j\left(-\sum_I x_{ij} + \sum_K y_{jk}\right) = 0 \qquad \text{for } j\varepsilon J \qquad (10)$$

$$\sum_j y_{jk} + t_k = b_k \qquad \text{and} \qquad w_k t_k = 0 \qquad \text{for } k\varepsilon K \qquad (11)$$

$$\sum_k \alpha_{ik} \le A_i \qquad \text{and} \qquad (A_i - \sum_k \alpha_{ik})\, q_i = 0 \quad \text{for } i\varepsilon I \qquad (12)$$

Relations (9) state that the total sales of a given supplier cannot exceed his available supply. But if they fall short of the available supply, the auction price fetched by this supplier must have dropped to zero. Relations (10) state that a wholesaler cannot in any given time period sell more than what he purchases (no account of inventories is made in this one-period model). If he does not sell everything he has on hand, the auction price of his goods must have fallen to zero. Relations (11) state that the total sales of all brands to a given consumer cannot exceed his demand. But if they do fall short of his demand, his surplus must vanish. Relations (12) state that the total advertising expenditure of each supplier cannot exceed his available advertising budget. But if they fall short of the budget, the imputed value of one additional dollar of advertising is zero.

Further,

$$u_i - v_j \ge - c_{ij} - h_j \quad \text{and } x_{ij}\,(u_i - v_j + c_{ij} + h_j) = 0 \qquad \text{for } i\varepsilon I,\ j\varepsilon J \qquad (13)$$

$$v_j + w_k \ge d_{jk} + d_{jk}'\alpha_{jk} - e_{jk} \quad \text{and } y_{jk}\,(v_j + w_k - d_{jk} - d_{jk}'\,\alpha_{jk} + e_{jk}) = 0 \text{ for } j\varepsilon J, k\varepsilon K \qquad (14)$$

$$q_j \ge d_{jk}'y_{jk} \qquad \text{and } \alpha_{jk}(q_j - d_{jk}'y_{jk}) = 0 \quad \text{for } j\varepsilon J, k\varepsilon K \qquad (15)$$

$$u_i \ge s_i \qquad \text{and } z_i\,(u_i - s_i) = 0 \qquad \text{for } i\varepsilon I \qquad (16)$$

$$w_k \ge - p_k \qquad \text{and } t_k(w_k + p_k) = 0 \qquad \text{for } k\varepsilon K \qquad (17)$$

Relations (13) express the condition that the imputed enhancement of value along any arc (i,j) of the transportation system can never exceed the unit transportation cost on the arc plus the packaging cost. Furthrmore, if there is a positive flow along the arc, the imputed enhancement of value on that arc equals the unit transportation plus packaging costs. Relations (14) state that the bid of a consumer can never exceed the sum of the market price, his surplus, and the unit transportation cost from a wholesaler to that customer. Moreover, if the consumer buys a positive quantity from that wholesaler, the bid must exactly equal the sum of the market price, his surplus, plus the unit transportation cost. That, of course, is just the definition of the buyer surplus. Relations (15) state that the imputed value of one additional dollar of advertising cannot fall short of the total bid value that such a dollar produces. Moreover, if advertising is positive, the imputed value must equal that bid value. Relations (16) state that the auction price cannot fall short of the reservation price of the supplier. If a supplier returns from the auction with some unsold quantity, the auction price must have dropped to the reservation price. Relations (17)

state that the surplus of a consumer is greater than or equal to the negative of the value he places on not buying one unit of the good. In the case that he is unable to buy all he wants, his surplus is equal to the negative of that value.

Finally, there are the following nonnegativity conditions:

$$x_{ij}, y_{jk}, \alpha_{jk}, z_i, t_k \geq 0 \qquad \text{for } i\varepsilon I, j\varepsilon J, k\varepsilon K$$
$$v_j \geq 0 \qquad \text{for } j\varepsilon J \qquad (18)$$

The unknowns u_i, $i\varepsilon I$ and w_k, $k\varepsilon K$ are unrestricted in sign.

6. A Numerical Example with Transportation and Distribution.

Consider an economy in which there are 3 suppliers of coffee beans, 2 wholesale roasters, and 4 final consumers. Suppose that the quality of all of the coffee beans for sale by the suppliers is identical (say Columbian coffee) but that each roaster possesses a unique roasting technique that lends a characteristic flavor to the finished product. The supplies, demands, unit transportation costs, reservation prices, packaging costs, and bids are given next.

The supplier to roaster data are:

Supplier	Supplies	Reservation price	Transportation costs to roaster 1	Transportation costs to roaster 2
1	80	2.5	2	4
2	100	3.25	5	3
3	60	1.95	4	5

The roaster to consumer data are:

Roaster	Advertising budget	Roasting costs	Transportation costs to consumer 1	Transportation costs to consumer 2	Transportation costs to consumer 3	Transportation costs to consumer 4
1	10	2.5	5	4	6	3
2	10	3.25	6	3	8	4

The bids in the absence of advertising, and the demands of the consumers are exhibited in the next table (below).

Roaster	Bid by consumer 1	Bid by consumer 2	Bid by consumer 3	Bid by consumer 4
1	22	18	21	25
2	25	23	22	20
Demands	30	60	80	70

Finally, we list the response of the bid prices to each dollar of advertising expenditure.

Advertising by	Additional bid by consumer 1	Additional bid by consumer 2	Additional bid by consumer 3	Additional bid by consumer 4
roaster 1	0.5	0.5	0.5	0.5
raster 2	0.1	0.1	0.1	0.1

All consumer surpluses are required to be nonnegative ($p_k = 0$ for all k).

The solution is listed in the boxes below. The solution advertising expenditures are exhibited first. Roaster 1 spends his entire advertising budget targeting consumer 2; roaster 2 spends his budget targeting consumer 3.

Advertising by roaster	targeting consumer 1	targeting consumer 2	targeting consumer 3	targeting consumer 4
1	0	0	10	0
2	0	10	0	0

The resulting quantities sold are listed next. Each supplier sells all his supplies. Each consumer fills his demands. Supplier to roaster shipping amounts :

From supplier	to roaster 1	to roaster 2
1	80	0
2	10	90
3	60	0

and roaster to consumer shipping amounts:

From roaster	to consumer 1	to consumer 2	to consumer 3	to consumer 4
1	0	0	80	70
2	30	60	0	0

Note that each consumer buys from one single roaster.

The solution bids (including the responses to advertising) are listed below. As a result of the advertising, the bid received by roaster 1 from consumer 3 has now been raised from 21 to 26. The bid received by roaster 2 from consumer 3 has been raised from 23 to 24.

Bids placed on goods offered by roaster	Bid by consumer 1	Bid by consumer 2	Bid by consumer 3	Bid by consumer 4
1	22	18	26	25
2	25	24	22	20

As explained, the solution for the auction prices depends on the perturbations applied to the supplies and demands. In the tables below, Solution 1 means the MinSeller-MaxBuyer solution (obtained by perturbing supplies) , Solution 2 means the MaxSeller-MinBuyer solution (obtained by perturbing demands) , and FDS means the fair division solution.

The supplier prices and surpluses (above the reservation prices) are:

Solution	Supplier price 1	Supplier price 2	Supplier price 3	Surplus of supplier 1	Surplus of supplier 2	Surplus of supplier 3
1	6.25	3.25	4.25	3.75	0	2.3
2	15.50	12.5	13.5	13	8.75	11.55
FDS	10.67	7.67	8.67	8.37	4.37	7.93

and the roaster prices are:

Solution	Roaster price 1	Roaster price 2
1	10.75	9.50
2	20.00	18.75
FDS	15.37	14.12

Finally, the consumer surpluses were:

Solution	Surplus of consumer 1	Surplus of consumer 2	Surplus of consumer 3	Surplus of consumer 4
1	9.50	11.50	9.25	11.25
2	0.25	2.25	0	2.00
FDS	4.87	6.87	4.67	6.67

Note that the smallest suppliers' surplus in the MinSeller-MaxBuyer solution is brought down to zero; in the MaxSeller-MinBuyer solution, the smallest consumers' surplus is brought down to zero.

7. Solution of Subproblem Defined by Coalitions

In [7] we observed that given our heterogeneous goods model it was easily possible to evaluate the value of a given coalition of players by just limiting someof the action possibilities in the model. This is similar, but not identical to, a "controlled programming problem" in the sense of Dubey and Shapley [1].

Here we show an example of the same idea by choosing a coalition of the players in Section 6. Consider the coalition S = {supplier 1, supplier 3, Buyer 2, Buyer 3}. To solve the subproblem defined by the coalition we make $a_2 = 0$, $b_1 = 0$, and $b_4 = 0$. We left the advertising budgets of the roasters to be the same as before. When the new problem was solved, we found the roaster to consumer bids were altered by the resulting heavy advertising campaign to be:

	Consumer 1	Consumer 4
Roaster 1	18	26
Roaster 2	24	22

Note that the entry in the lower left hand corner is 24 instead of 23 as it was previously, and also the entry in the upper right hand corner has been increased from 21 to 26 all due to advertising.

The supplier prices and surpluses are:

Solution	Supplier 1	Supplier 3	Surplus of 1	Surplus of 3
1	2.95	1.95	.45	0
2	14.75	12.75	12.25	10.8
FDS	8.85	7.35	6.35	5.4

The roaster prices were:

Solution	Roaster 1	Roaster 2
1	7.45	10.2
2	9.25	21.0
FDS	13.35	15.6

Finally, the consumer surpluses were:

Solution	Consumer 2	Consumer 2
1	11.8	12.55
2	0	.75
FDS	5.4	6.65

It is obvious that by making use of such coalitions one could develop a model containing several manufacturing firms, some of which are vertically integrated, some horizontally integrated, and others just independent units. The ease with which one can set up and solve the resulting models should make this approach be useful for studying business strategy questions.

8. Concluding Remarks.

An economic model is never the same as reality itself. It is a simplified portrayal, a simile. As economic science has progressed, modeling has become more sophisticated, and-- hopefully -- more realistic. The model of a free competitive market remains a cornerstone in economic analysis. And yet, its relevance in the modern economy has shrunk as fewer and fewer goods satisfy the classical assumptions that the good transacted be homogeneous and that sellers and buyers alike can be viewed as price takers.

In these pages, we have shown how it is possible to recast the assumptions of competition and free markets in a vestige that retains the concept of free market formation and yet permits the goods to be heterogeneous, as are most goods sold under proprietary brand names. The new market paradigm proposed is that of the sealed bid auction.

Before the auction, the prospective buyers of various brands inspect the wares that have been put up for sale and submit written (and sealed) bids on each brand. A buyer finding one brand more appealing than another, will also offer a higher bid for it. Thus, the diversity of the goods exchanged at the auction is reflected in the diversity of the bids.

Even if no formal sealed bid auction is conducted, it is suggested that the market formation may be described as if such an auction took place. Each buyer is assumed to behave as if a full range of bid prices for all brands were submitted to an auctioneer. Indirectly, those bid prices would reflect the subjective estimations of each buyer of the qualities and attractiveness of a brand. Observing the behavior of the buyers in the marketplace, the analyst would be able to calculate the difference between a buyer's bid price and the market price. In the case of a "good buy", the consumers surplus comes out as large, positive. In the case of marginal purchases that the buyer just barely finds worthwhile, the surplus is zero.

We propose that the sealed bid auction together with transportation systems provides a modeling technique with which it is easy to set up and solve economic models of an industry in which there are a variety of organizational forms of business firms competing with each other in the production, distribution, and marketing of a given product. It should be possible, by using this approach, to make assertions concerning the suitability of a kind of an organization to compete against other kinds of organizations.

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